Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

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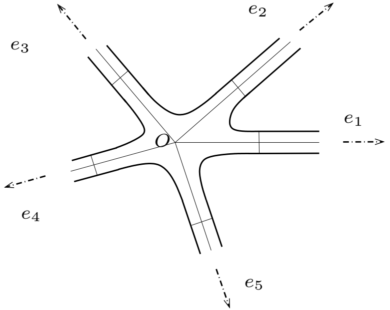
Setting: a thick version of a network

Consider a domain $\Omega \subset \mathbb{R}^2$:

- star-shaped w.r.t. the origin O
- $\partial\Omega$ is smooth
- Far enough from the origin O, Ω coincides with the union of N non-intersecting semi-infinite strips directed by the vectors e_i, $i = 1, \ldots, N$

The domain Ω



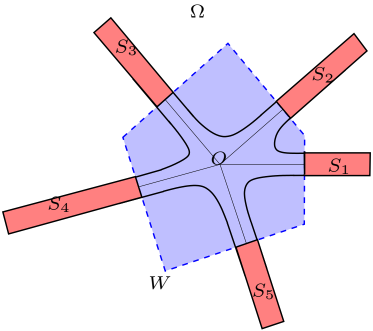
Setting: a more precise definition

- for some $r_0 > 0$, let W be the polygonal set

$$W = \{x \in \mathbb{R}^2 : \quad x \cdot e_i \leq r_0, \ \forall i = 1, \dots, N\}$$

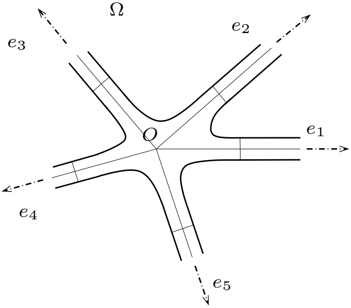
- Then $\Omega \setminus W = \bigcup_{i=1}^{N} S_i$, where S_i is the half-strip

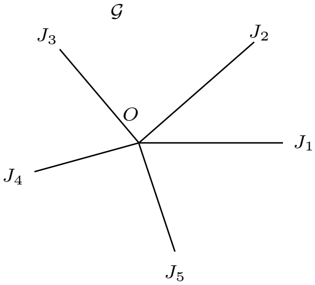
$$S_i = \left\{ x_i e_i + x_i^{\perp} e_i^{\perp}, \quad x_i > r_0, \ |x_i^{\perp}| < 1 \right\}$$



$\Omega_\epsilon = \epsilon\Omega$ “tends” to a network $\mathcal{G}$ as $\epsilon \to 0$

$$
\begin{aligned}
\mathcal{G} &= \{O\} \cup \bigcup_{i=1}^{N} J_i, \\
J_i &= \{x_i e_i, \;\; x_i > 0\}.
\end{aligned}
$$





State constrained control problems in $\overline{\Omega_\epsilon}$

$$u_\epsilon(x) = \inf_\alpha \int_0^\infty \ell_\epsilon\Big(y_\epsilon(t;x,\alpha), \alpha(t)\Big) e^{-\lambda t} dt$$

subject to

$$\left\{ \begin{array}{rcll} \dot{y}_\epsilon(t;x,\alpha) & = & \alpha(t), & \quad \text{a.a. } t > 0, \\ y_\epsilon(0;x,\alpha) & = & x, & \end{array} \right.$$

with $\alpha: \mathbb{R}_+ \to A$ measurable, under the state constraint

$$y_\epsilon(t) \in \overline{\Omega}_\epsilon, \quad \forall t \geq 0.$$

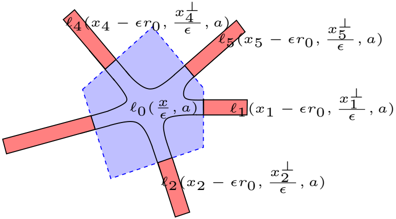
Controlability : A is a compact subset of $\mathbb{R}^2$ such that $B(0,r) \subset A$ for some $r > 0$.

Assumptions on the running costs

- The function $\ell_\epsilon : \Omega_\epsilon \to \mathbb{R}$ is bounded and continuous
- $$\ell_\epsilon(x,a) = \begin{cases} \ell_i\left(x_i - \epsilon r_0, \dfrac{x_i^\perp}{\epsilon}, a\right) & \text{in } \epsilon S_i \\ \ell_0\left(\dfrac{x}{\epsilon}, a\right) & \text{in } \epsilon W \cap \Omega_\epsilon \end{cases}$$

$\ell_i : [0,+\infty) \times [-1,1] \times A \to \mathbb{R}$

$\ell_0 : (W \cap \Omega) \times A \to \mathbb{R}$



ℓ_i and ℓ_0 match properly in order to ensure the continuity of ℓ_ϵ

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Remark

In ϵS_i, ℓ_ϵ varies slowly w.r.t. x_i and fast w.r.t. $x_i^\perp$.

- It is not restrictive to assume that $\ell_0 \geq 0$.

The value function u_ϵ

u_ϵ is bounded uniformly with respect to ϵ, continuous, and is the unique viscosity solution of

$$\begin{aligned} \lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\geq 0 \qquad \text{in } \overline{\Omega}_\epsilon, \\ \lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\leq 0 \qquad \text{in } \Omega_\epsilon, \end{aligned}$$

where

$$H_\epsilon(x,p) = \max_{a \in A} \Big(-p \cdot a - \ell_\epsilon(x,a) \Big).$$

Questions

- Asymptotic behavior of u_ϵ as $\epsilon \to 0$?
- Can we define an effective problem in $\mathcal{G}$?

Background and additional new difficulties

- Existing results:
  - Singularly perturbed control problems in thin domains around smooth manifolds: Bensoussan, Alvarez-Bardi, Arztein-Gaitsgory, Gaitsgory-Leisarowitz, Terrone,...
  - Comparison for viscosity solutions on networks: YA-Oudet-Tchou and Imbert-Monneau
  - Very recent results of Lions-Souganidis on homogenization of HJB equations with defects

- Several additional new difficuties:
  - Identification of the effective problem
  - Will the effective problem keep track of ℓ_0 near O? How?
  - In the perturbed test-function method of Evans, we need to construct correctors in unbounded domains.

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \to \mathbb{R}$ of

$$\left\{ \begin{array}{rcll} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) & = & 0 & x \in J_i, \\ \\ \lambda u(O) + \max\left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0))\right) & = & 0, & \end{array} \right.$$

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with

$$\overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left(\int_{[-1,1]\times A} \Big(-p_i a_i - \ell_i(x_i, y, a)\Big) d\mu(y, a) \right)$$

$\mathcal{Z}_i$ is a compact and convex set of Radon probability measures on $[-1, 1] \times A$

$\mathcal{Z}_i$: limiting relaxed controls.

Main result

Theorem

Under a further technical assumption, u_ϵ converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \to \mathbb{R}$ of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max\left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0))\right) = 0, \end{cases}$$

with

$$-E : \text{ an effective cost at the junction}$$

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$$\left\{ \begin{array}{rcll} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) & = & 0 & x \in J_i, \\ \\ \lambda u(O) + \max\left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0))\right) & = & 0, & \end{array} \right.$$

with $\qquad \overline{H}(O, p_1, \dots, p_N) := \max_{i=1,\dots,N} \overline{H}_i^+(0, p_i),$

$$\overline{H}_i^+(0, p_i) = \sup_{\mu \in \mathcal{Z}_i^+} \left(\int_{[-1,1]\times A} \Big(-p_i a_i - \ell_i(0, y, a) \Big) d\mu(y, a) \right),$$

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1]\times A} a_i d\mu(y, a) \geq 0 \right\}.$$

Hamilton-Jacobi on the network (1/2)

What is the meaning of

$$\left\{\begin{array}{l} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 \quad x \in J_i, \\ \\ \lambda u(O) + \max\left(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0))\right) = 0 \quad ? \end{array}\right. \tag{1}$$

See YA-Camilli-Cutrí-Tchou(2013), Imbert-Monneau- -Zidani(2013), YA-Oudet-Tchou(2014), and Imbert- -Monneau(2014).

Definition (Test functions)

- $\phi : \mathcal{G} \to \mathbb{R}$ is an admissible test-function if
  - ϕ is continuous on $\mathcal{G}$
  - for any $j \in 1, \dots, N$, $\phi|_{\overline{J}_j} \in \mathcal{C}^1(\overline{J}_j)$
- $\mathcal{R}(\mathcal{G})$: set of the admissible test-functions

- An usc function $u : \mathcal{G} \to \mathbb{R}$ is a sub-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local maximum in x, then

$$\lambda u(x) + \overline{H}_i(x, \tfrac{d\phi}{dx_i}(x)) \le 0 \qquad \text{if } x \in J_i,$$
$$\lambda u(O) + \max\left(E, \overline{H}(\tfrac{d\phi}{dx_1}(O), \dots, \tfrac{d\phi}{dx_N}(O))\right) \le 0.$$

- A lsc function $u : \mathcal{G} \to \mathbb{R}$ is a super-solution of (1) if for any $x \in \mathcal{G}$ and $\phi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \phi$ has a local minimum in x, then

$$\lambda u(x) + \overline{H}_i(x, \tfrac{d\phi}{dx_i}(x)) \ge 0 \qquad \text{if } x \in J_i,$$
$$\lambda u(O) + \max\left(E, \overline{H}(\tfrac{d\phi}{dx_1}(O), \dots, \tfrac{d\phi}{dx_N}(O))\right) \ge 0.$$

The effective Hamiltonians away from the junctions

Notation: for $x \geq 0, y \in [-1,1], p \in \mathbb{R}^2$,

$$H_i(x,y,p) = \max_{a \in A}(-p \cdot a - \ell_i(x,y,a))$$

Theorem (Alvarez-Bardi 2000)

$\forall x_i \geq 0$, $\forall p_i \in \mathbb{R}$, there is a unique $\overline{H}_i(x_i,p_i)$ s.t. the cell problem

$$\begin{aligned} &H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^{\perp}) \geq \overline{H}_i(x_i,p_i) \qquad y \in [-1,1],\\ &H_i(x_i, y, p_i e_i + D_y \chi_i(y)) e_i^{\perp} \leq \overline{H}_i(x_i,p_i) \qquad y \in (-1,1) \end{aligned}$$

has a viscosity solution $\chi_i \in \mathrm{Lip}([-1,1])$.

Limiting relaxed controls and limit control problem

For $s > 0$, the *occupational measure* μ_s generated by $(y(t), \alpha(t))$ is the Radon probability measure defined on $[-1, 1] \times A$ by

$$\mu_s = \frac{1}{s} \int_0^s \delta_{(y(t),\alpha(t))} dt$$

where $\delta_{(y(t),\alpha(t))}$ is the Dirac mass concentrated at $(y(t), \alpha(t))$.

$\mathcal{Z}(s; i, y_0)$: set of the occupational measures generated by the trajectories $(y(t), \alpha(t))$ up to time s.

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Theorem (Gaitsgory and Leisarowitz (1999))

There exists a set $\mathcal{Z}_i \subset \mathcal{P}([-1, 1] \times A)$ *s.t*

$$\forall y_0 \in [-1, 1], \qquad \lim_{s \to \infty} \pi_H(\mathcal{Z}(s; i, y_0), \mathcal{Z}_i) = 0,$$

(π_H : Prokhoroff distance).
$\mathcal{Z}_i$ is convex and compact for the weak- topology.*

By using results from Terrone-2011 and Alvarez-Bardi-2000

- $$\mathcal{Z}_i \subset \left\{ \mu \in \mathcal{P}([-1,1]\times A),\ \int_{[-1,1]\times A} a_i^{\perp} d\mu(y,a) = 0 \right\}$$

- $\mathcal{Z}_i$ coincides with the set of limiting relaxed controls, i.e.

$$\overline{H}_i(x_i,p_i) = \sup_{\mu\in\mathcal{Z}_i} \left(-p_i \int_{[-1,1]\times A} a_i d\mu(y,a) - \int_{[-1,1]\times A} \ell_i(x_i,y,a) d\mu(y,a) \right)$$

Interpretation

Away from the junction, the effective Hamiltonian corresponds to a control problem with controls in $\mathcal{Z}_i$.

The effective Hamiltonian at the junction

Definition

The effective Hamiltonian at the junction is $\max(E, \overline{H}(O,\cdot))$ where E is a suitable constant to be defined and

$$\overline{H}(O, p_1, \dots, p_N) := \max_{i=1,\dots,N} \overline{H}_i^+(0, p_i)$$

$$\overline{H}_i^+(0, p_i)$$
$$= \sup_{\mu \in \mathcal{Z}_i^+} \left(-p_i \int_{[-1,1]\times A} a_i d\mu(y,a) - \int_{[-1,1]\times A} \ell_i(0,y,a) d\mu(y,a) \right)$$

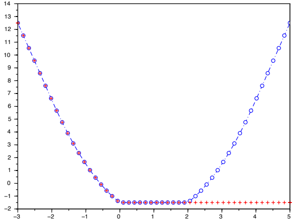
and

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1]\times A} a_i d\mu(y,a) \geq 0 \right\}$$

Lemma: links between $\overline{H}_i(0,p)$ and $\overline{H}_i^+(0,p)$

$$\mathcal{Z}_i^0 := \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1]\times A} a_i d\mu(y,a) = 0 \right\}$$

$$\text{argmin}\overline{H}_i(0,\cdot) = \left[\ \underline{p}_i\ ,\ \overline{p}_i\ \right]$$



The graphs of $p \mapsto \overline{H}_i(0,p)$ and of $p \mapsto \overline{H}_i^+(0,p)$

- $p < \underline{p}_i \Rightarrow \overline{H}_i(0,p) = \overline{H}_i^+(0,p)$ is achieved for $\mu \in \mathcal{Z}_i^+ \backslash \mathcal{Z}_i^0$
- $\underline{p}_i < p < \overline{p}_i \Rightarrow \overline{H}_i(0,p) = \overline{H}_i^+(0,p)$ is achieved for $\mu \in \mathcal{Z}_i^0$
- $p > \overline{p}_i \Rightarrow \overline{H}_i(0,p) > \overline{H}_i^+(0,p) = \min \overline{H}_i(0,\cdot)$

The constant E

Zoom near the junction point O

We extend the function ℓ_0 to the whole domain $\overline{\Omega}$ by setting

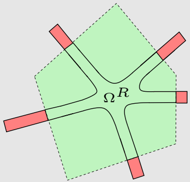
$$\ell_0(z,a) = \ell_i(0, z_i^\perp, a), \quad \text{if } z_i \geq r_0,\ |z_i^\perp| \leq 1.$$

Ergodic constants in bounded subdomains

Define $\Omega^R := \Omega \cap W_R$ and the ergodic constant E^R, which is the unique number s.t.

$$\begin{aligned} H_0(z, Dw^R(z)) &\geq E^R \qquad \text{in } \overline{\Omega}^R, \\ H_0(z, Dw^R(z)) &\leq E^R \qquad \text{in } \Omega^R, \end{aligned}$$

has a viscosity solution w^R.



The constant E

Lemma

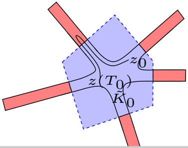
$\exists C > 0 \;\; s.t.$

$$\begin{aligned} |E^R| \le C & \qquad \forall R, \\ \|Dw^R\|_\infty \le C & \qquad \forall R, \\ R \le R' \qquad \Rightarrow & \qquad E^R \le E^{R'}. \end{aligned}$$

Definition

$$E := \lim_{R\to\infty} E^R.$$

The constant E



Theorem

$\exists C > 0 \ s.t. \quad \forall z_0 \in \widetilde{K}_0, \forall \alpha \in \mathcal{A}_{z_0}, \forall T > 0,$

if $z(s) := z_0 + \int_0^s \alpha(\theta) d\theta$ *satisfies* $z(T) \in \widetilde{K}_0$, *then*

$$\int_0^T \ell_0(z(s), \alpha(s)) ds \geq -ET - C.$$

Lemma

Under the assumptions, $\forall i = 1, \ldots, N, \qquad \min_{p_i} \overline{H}_i(0, p_i) \leq E.$

Proof of the convergence result

The main step is to prove the following theorem

Theorem

Consider the relaxed semilimits, of u_ϵ: for all $x \in \mathcal{G}$,

$$\underline{u}(x) = \liminf_{\epsilon\to 0+, x'\to x, x'\in\Omega_\epsilon} u_\epsilon(x'); \quad \overline{u}(x) = \limsup_{\epsilon\to 0+, x'\to x, x'\in\Omega_\epsilon} u_\epsilon(x').$$

Then, under the assumptions, $\underline{u}$ is a bounded supersolution of

$$\begin{array}{rcll} \lambda\underline{u}(x) + \overline{H}(x, D\underline{u}(x)) & \geq & 0, & \text{if } x \in \mathcal{G}\backslash\{O\}, \\ \lambda\underline{u}(O) + \max(E, \overline{H}(O, D\underline{u}(O))) & \geq & 0, & \text{if } x = O, \end{array}$$

and $\overline{u}$ is a bounded subsolution of

$$\begin{array}{rcll} \lambda\overline{u}(x) + \overline{H}(x, D\overline{u}(x)) & \leq & 0, & \text{if } x \in \mathcal{G}\backslash\{O\}, \\ \lambda\overline{u}(O) + \max(E, \overline{H}(O, D\overline{u}(O))) & \leq & 0, & \text{if } x = O. \end{array}$$

then use comparison results on $\mathcal{G}$, [AOT2014] or [IM2014]

Some ideas of the proof

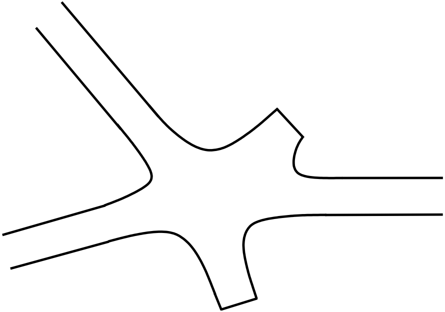
- Not standard only at the junction
  - Try to use Evans' method of perturbed test-functions
  - The test-functions ϕ have N slopes at O : $p = (p_1, \dots, p_N)$
  - Evans' method requires the construction of bounded correctors in unbounded domains, (ideally Ω)
  - Bounded correctors may not exist in the full domain Ω, because, given a set of slopes at O, i.e. $p = (p_1, \dots, p_N)$,

    $$\overline{H}(O, p_1, \dots, p_n) = \max_{i=1,\dots,N} \overline{H}_i^+(0, p_i)$$

    is achieved by $i \in \mathcal{I}(p) \subset \{1, \dots, N\}$. If $j \notin \mathcal{I}(p)$, then the optimal trajectories starting from $z \in S_j$ should leave S_j: this leads to the fact that the corrector should not be bounded when $|z| \to \infty$, $z \in S_j$.
  - Hence, we construct the correctors in subdomain obtained by truncating the half-strips S_j if $j \notin \mathcal{I}(p)$.

The corrector (when $\overline{H}(O, p_1, \dots, p_N) > E$)

The truncated domain Ω_p



Theorem

For $p \in \mathbb{R}^N$: $\overline{H}(O,p) > E$. Let ψ_p be a smooth function such that $D\psi_p = p_i e_i$ in S_i. Under the assumptions, $\exists \chi_p$, a bounded and Lipschitz viscosity solution of

$$
\begin{aligned}
H_0(z, D\psi_p + D\chi_p) - \overline{H}(O,p) &\geq 0 \qquad \text{in } \overline{\Omega}_p,\\
H_0(z, D\psi_p + D\chi_p) - \overline{H}(O,p) &\leq 0 \qquad \text{in } \Omega_p.
\end{aligned}
$$

The further assumption

Assumption

1) For any real number p_i such that $p_i < \underline{p}_i$,

there exist two constants $L_i \geq 0$ and $C_i > 0$ such that

$\forall y_0 \in [-1, 1]$, $\forall t > 0$, *there exists a control law $\tilde{\alpha} \in \mathcal{A}_{i,y_0}$ with*

$$\int_0^s \tilde{\alpha}_i(\tau) d\tau \geq -L_i, \quad \forall 0 \leq s \leq t,$$

$$\int_0^t (p_i \tilde{\alpha}_i(s) + \ell_i(0, y(s), \tilde{\alpha}(s)))\, ds \leq v_i(0, p_i, y_0, t) + C_i,$$

where $y(t) = y_0 + \int_0^t \tilde{\alpha}_i^{\perp}(s) ds$.

Recall that

$$v_i(x_i, p_i, y_0, t) = \inf_{\alpha \in \mathcal{A}_{i,y_0}} \left\{ \int_0^t p_i \alpha_i(s) + \ell_i(x_i, y(s), \alpha(s)) ds \right\}$$